# Matrices and Systems of Equations

(Section 10.1 and 10.2)





#### **Matrices!**

#### Matrix-a rectangular array of real numbers

Matrices are in form  $m \times n$  (m by n), where m is the number of rows and n is the number of columns.

Entries in the *i*th row and *j*th column are denoted by  $a_{jj}$  (so  $a_{32}$  is in the  $3^{rd}$  row and  $2^{nd}$  column)





#### **Example 1**

Determine the order of each matrix:

a) 
$$\begin{bmatrix} 3 & 5 \\ 14 & -9 \end{bmatrix}$$

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$$\begin{bmatrix} 3 & 5 \\ 14 & -9 \end{bmatrix}$$
 c)  $\begin{bmatrix} 18 & 7 & 1 \\ -64 & 2 & 0 \end{bmatrix}$ 

# Augmented and Coefficient Matrices



- Augmented Matrix-a matrix derived from a system of linear equations
- <u>Coefficient Matrix-a</u> matrix derived from the coefficients of the system (not including the constant terms)



### Example 2

Write this system as an augmented matrix and a coefficient matrix

$$x+2y-2z = 5$$
$$-x-3y+z = 1$$
$$2x + 3z = 4$$



### **Example 2 Continued**

#### Answer:

$$\begin{bmatrix} 1 & 2 & -2 & \vdots & 5 \\ -1 & -3 & 1 & \vdots & 1 \\ 2 & 0 & 3 & \vdots & 4 \end{bmatrix} \xrightarrow{\text{IMPORTANT: Notice the } 0 \text{ used as a placeholder}}$$

$$\begin{bmatrix} 1 & 2 & -2 \\ -1 & -3 & 1 \\ 2 & 0 & 3 \end{bmatrix} \xrightarrow{\text{Coefficient Matrix}}$$

#### **Elementary Row Operations...**



...used on an augmented matrix of a given system of linear equations to produce a new augmented matrix corresponding to a new (but equivalent) system of linear equations.

#### The Elementary Row Operations are:

- Interchange two rows
- 2. Multiply a row by a nonzero constant
- 3. Add a multiple of a row to another row

# So basically...



...elementary row operations are things you can do to a matrix to produce a **row- equivalent** matrix (which will help to isolate a variable)

# Gaussian Elimination and Row-Echelon Form



Gaussian Elimination is a way of using the elementary row operations to isolate a variable in an augmented matrix.

1	3	-2	:	6
2	- 2	0	÷	- 4
1	5	3	:	2
0	2	3	:	5
0	0	3	:	9

A row like this would translate into 3z=9, from which we could find that z=3

#### **Row-Echelon Form**

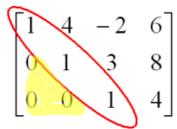


A matrix in Row-Echelon Form has these properties

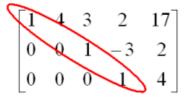
- All rows consisting entirely of zeros are at the bottom of the matrix.
- For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a **leading** 1)
- For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

# **Matrices in Row-Echelon Form**





The diagonal row of 1s is the row of leading ones, and all spots before the leading ones are zeros.



Although the leading ones are not in a diagonal, the 2<sup>nd</sup> row's leading one is farther to the right than the 1<sup>st</sup> row's, so it is still in row-echelon form

# Gaussian Elimination With Back Substitution



Use the elementary row operations to put an augmented matrix into Row-Echelon form, and then take the isolated variable and plug it back into the system of equations.



### Example 3

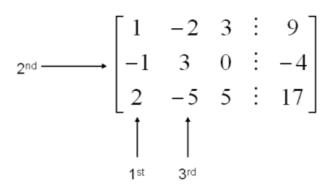
Solve this system of equations using an Gaussian Elimination with Back-Substitution.

$$x-2y+3z = 9$$
$$-x+3y = -4$$
$$2x-5y+5z = 17$$



#### The Easy Way

Once the system is written as an augmented matrix, concentrate on making only 3 spots into zeros.





#### Step 1

Add 2 times the second row to the third row.

$$\begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ -1 & 3 & 0 & \vdots & -4 \\ 2 & -5 & 5 & \vdots & 17 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ -1 & 3 & 0 & \vdots & -4 \\ 0 & 1 & 5 & \vdots & 9 \end{bmatrix}$$



#### Step 2

Add the first row and second row together.



### Step 3

Add -1 times the second row to the third row.

$$\begin{bmatrix}
1 & -2 & 3 & \vdots & 9 \\
0 & 1 & 3 & \vdots & 5 \\
0 & 1 & 5 & \vdots & 9
\end{bmatrix}$$

$$= \begin{bmatrix}
1 & -2 & 3 & \vdots & 9 \\
0 & 1 & 3 & \vdots & 5 \\
0 & 0 & 2 & \vdots & 4
\end{bmatrix}$$



#### **Back-Substitution**

$$\begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ 0 & 1 & 3 & \vdots & 5 \\ 0 & 0 & 2 & \vdots & 4 \end{bmatrix} \longrightarrow z = 2$$

$$y+3z = 5$$
$$y+3(2) = 5$$
$$y = -1$$

$$x-2y+3z = 9$$
$$x-2(-1)+3(2) = 9$$
$$x = 1$$

#### **Systems With Infinite Solutions**



A matrix in which...

- None of the variables are isolated
- There is only one non-zero row

...has an infinite number of solutions.

Ex: 
$$\begin{bmatrix} 2 & 1 & \vdots & 3 \\ 0 & 0 & \vdots & 0 \\ 0 & 0 & \vdots & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 5 & \vdots & 2 \\ 0 & 1 & -3 & \vdots & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 5 & \vdots & 2 \\ 0 & 1 & -3 & \vdots & -1 \end{bmatrix}$$

# Solving Systems With Infinite Solutions



$$\begin{bmatrix} 1 & 0 & 5 & \vdots & 2 \\ 0 & 1 & -3 & \vdots & -1 \end{bmatrix} \longrightarrow \begin{array}{c} x + 5z = 2 \\ y - 3z = -1 \end{array}$$

Let z=a.

x+5a=2

x = -5a + 2

y-3a=-1

y=3a-1

Solutions:

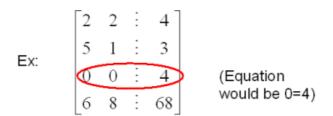
(-5a+2, 3a-1, a)



# **Systems With No Solution**

A system has no solution when

a row has all zeros except for the last entry



### Gauss Jordan elimination:

$$-x + y - z = -14$$
  
 $2x - y + z = 21$   
 $3x 2y + z = 19$