Matrices and Systems of Equations

(Section 10.1 and 10.2)

Matrices!

Matrix-a rectangular array of real numbers

Row (horizontal) → ; Column (vertical) ↓

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{3n} \\ a_{m1} & a_{m2} & a_{m3} & a_{mn} \end{bmatrix}$$

Matrices are in form m x n (m by n), where m is the number of rows and n is the number of columns.

Order : R×C

Entries in the *i*th row and *j*th column are denoted by a_{ij} (so a_{32} is in the 3rd row and 2nd column)

Example 1

Determine the order of each matrix:

a) $\begin{bmatrix} 3 & 5 \\ 14 & -9 \end{bmatrix}$

2x2

b) $\begin{bmatrix} 16 & 7 & 1 & 17 \end{bmatrix}$

1x4

c) $\begin{bmatrix} 18 & 7 & 1 \\ -64 & 2 & 0 \end{bmatrix}$

2x3

$$2x + 2y - 6z = -6$$

$$x + y - 2z = 0$$

$$-2x - y + 8z = 19$$

	x	y	z		c
Eq.1	2	2	-6	⋮	-6
Eq.2	1	1	-2	⋮	0
Eq3	-2	-1	8	⋮	19

Augmented and Coefficient Matrices

- Augmented Matrix-a matrix derived from a system of linear equations

- Coefficient Matrix-a matrix derived from the coefficients of the system (not including the constant terms)

Example 2

Write this system as an augmented matrix and a coefficient matrix

$$x + 2y - 2z = 5$$

$$-x - 3y + z = 1$$

$$2x \_\_\_\_ + 3z = 4$$

$$\left[\begin{array}{ccc:c} 1 & 2 & -2 & 5 \\ -1 & -3 & 1 & 1 \\ 2 & 0 & 3 & 4 \end{array}\right]$$

3 x 4

$$\begin{bmatrix} 1 & 2 & -2 \\ -1 & -3 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

3 x 3

square

Example 2 Continued

Answer:

$$\begin{bmatrix} 1 & 2 & -2 & \vdots & 5 \\ -1 & -3 & 1 & \vdots & 1 \\ 2 & 0 & 3 & \vdots & 4 \end{bmatrix}$$

← Augmented Matrix

IMPORTANT: Notice the 0 used as a placeholder

$$\begin{bmatrix} 1 & 2 & -2 \\ -1 & -3 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

← Coefficient Matrix

Elementary Row Operations…

…used on an augmented matrix of a given system of linear equations to produce a new augmented matrix corresponding to a new (**but equivalent**) system of linear equations.

The Elementary Row Operations are:

1. Interchange two rows
2. Multiply a row by a nonzero constant
3. Add a multiple of a row to another row

So basically…

…elementary row operations are things you can do to a matrix to produce a **row-equivalent** matrix (which will help to isolate a variable)

Gaussian Elimination and Row-Echelon Form

Gaussian Elimination is a way of using the elementary row operations to isolate a variable in an augmented matrix.

$$\begin{bmatrix} 1 & 3 & -2 & \vdots & 6 \\ 2 & -2 & 0 & \vdots & -4 \\ 1 & 5 & 3 & \vdots & 2 \\ 0 & 2 & 3 & \vdots & 5 \\ 0 & 0 & 3 & \vdots & 9 \end{bmatrix}$$

A row like this would translate into 3z=9, from which we could find that z=3

Row-Echelon Form

A matrix in Row-Echelon Form has these properties

1. All rows consisting entirely of zeros are at the bottom of the matrix.
2. For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a **leading 1**)
3. For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

Matrices in Row-Echelon Form

$$\begin{bmatrix} 1 & 4 & -2 & 6 \\ 0 & 1 & 3 & 8 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

The diagonal row of 1s is the row of leading ones, and all spots before the leading ones are zeros.

$$\begin{bmatrix} 1 & 4 & 3 & 2 & 17 \\ 0 & 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Although the leading ones are not in a diagonal, the 2nd row's leading one is farther to the right than the 1st row's, so it is still in row-echelon form

Gaussian Elimination With Back Substitution

Use the elementary row operations to put an augmented matrix into Row-Echelon form, and then take the isolated variable and plug it back into the system of equations.

Example 3

Solve this system of equations using an Gaussian Elimination with Back-Substitution.

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y \_\_ &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

The Easy Way

Once the system is written as an augmented matrix, concentrate on making only 3 spots into zeros.

$$\text{2nd} \longrightarrow \begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ -1 & 3 & 0 & \vdots & -4 \\ 2 & -5 & 5 & \vdots & 17 \end{bmatrix}$$

$$\uparrow \text{ 1st} \qquad \uparrow \text{ 3rd}$$

Step 1

Add 2 times the second row to the third row.

$$2R_2+R_3 \longrightarrow \begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ -1 & 3 & 0 & \vdots & -4 \\ 2 & -5 & 5 & \vdots & 17 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ -1 & 3 & 0 & \vdots & -4 \\ 0 & 1 & 5 & \vdots & 9 \end{bmatrix}$$

Step 2

Add the first row and second row together.

$$R_1+R_2 \longrightarrow \begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ -1 & 3 & 0 & \vdots & -4 \\ 0 & 1 & 5 & \vdots & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ 0 & 1 & 3 & \vdots & 5 \\ 0 & 1 & 5 & \vdots & 9 \end{bmatrix}$$

Step 3

Add -1 times the second row to the third row.

$$-R_2+R_3 \longrightarrow \begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ 0 & 1 & 3 & \vdots & 5 \\ 0 & 1 & 5 & \vdots & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ 0 & 1 & 3 & \vdots & 5 \\ 0 & 0 & 2 & \vdots & 4 \end{bmatrix}$$

Back-Substitution

$$\begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ 0 & 1 & 3 & \vdots & 5 \\ 0 & 0 & 2 & \vdots & 4 \end{bmatrix} \longrightarrow$$

$$2z = 4$$
$$z = 2$$

$$y + 3z = 5$$
$$y + 3(2) = 5$$
$$y = -1$$

$$x - 2y + 3z = 9$$
$$x - 2(-1) + 3(2) = 9$$
$$x = 1$$

Systems With Infinite Solutions

A matrix in which..

- None of the variables are isolated
- There is only one non-zero row

…has an infinite number of solutions.

Ex: $\begin{bmatrix} 2 & 1 & \vdots & 3 \\ 0 & 0 & \vdots & 0 \\ 0 & 0 & \vdots & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 5 & \vdots & 2 \\ 0 & 1 & -3 & \vdots & -1 \end{bmatrix}$

Solving Systems With Infinite Solutions

$$\begin{bmatrix} 1 & 0 & 5 & \vdots & 2 \\ 0 & 1 & -3 & \vdots & -1 \end{bmatrix} \longrightarrow \begin{aligned} x + 5z &= 2 \\ y - 3z &= -1 \end{aligned}$$

Let z=a.

x+5a=2

x=-5a+2

y-3a=-1

y=3a-1

Solutions:

(-5a+2, 3a-1, a)

Systems With No Solution

A system has no solution when

- a row has all zeros except for the last entry

Ex:
$$\begin{bmatrix} 2 & 2 & \vdots & 4 \\ 5 & 1 & \vdots & 3 \\ 0 & 0 & \vdots & 4 \\ 6 & 8 & \vdots & 68 \end{bmatrix}$$

(Equation would be 0=4)

Gauss Jordan elimination:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

reduced row echelon form

rref

x- 2y+3z=9
-x+3y   = -4
2x-5y+5z=17

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right] \xrightarrow{R1+R2} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 2 & -5 & 5 & 17 \end{array}\right]$$

$$2R1-R3 \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 1 & 1 & 1 \end{array}\right] \rightarrow R2-R3 \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

$$\frac{1}{2}R3 \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right] \rightarrow R2-3R3 \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$R1-3R3 \left[\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \rightarrow R1+2R2 \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$(1, -1, 2)$$

$$-x + y - z = -14$$
$$2x - y + z = 21$$
$$3x \quad 2y + z = 19$$